International Journal of Theoretical Physics, Vol. 44, No. 7, July 2005 (© 2005)
DOI: 10.1007/s10773-005-7069-4

State Property Systems and Orthogonality

Diederik Aerts[1,3] **and Didier Deses**[1,2,3]

Received August 14, 2004; accepted October 4, 2004

The structure of a state property system was introduced to formalize in a complete way the operational content of the Geneva–Brussels approach to the foundations of quantum mechanics (Aerts, D. *International Journal of Theoretical Physics*, **38**, 289–358, 1999; Aerts, D. in *Quantum Mechanics and the Nature of Reality*, Kluwer Academic; Aerts, D., Colebunders, E., van der Voorde, A., and van Steirteghem, B. *International Journal of Theoretical Physics*, **38**, 359–385, 1999), and the category of state property systems was proven to be equivalent to the category of closure spaces (Aerts, D., Colebunders, E., van der Voorde, A., and van Steirteghem, B., *International Journal of Theoretical Physics*, **38**, 359–385, 1999; Aerts, D., Colebunders, E., van der Voorde, A., and van Steirteghem, B., The construct of closure spaces as the amnestic modification of the physical theory of state property systems, *Applied Categorical Structures*, in press). The first axioms of standard quantum axiomatics (state determination and atomisticity) have been shown to be equivalent to the T_0 and T_1 axioms of closure spaces (van Steirteghem, B., *International Journal of Theoretical Physics*, **39**, 955, 2000; van der Voorde, A., *International Journal of Theoretical Physics*, **39**, 947–953, 2000; van der Voorde, A., *Separation Axioms in Extension Theory for Closure Spaces and Their Relevance to State Property Systems*, Doctoral Thesis, Brussels Free University, 2001), and classical properties to correspond to clopen sets, leading to a decomposition theorem into classical and purely nonclassical components for a general state property system (Aerts, D., van der Voorde, A., and Deses, D., *Journal of Electrical Engineering*, **52**, 18–21, 2001; Aerts, D., van der Voorde, A., and Deses, D. *International Journal of Theoretical Physics*; Aerts, D. and Deses, D., *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Computation, and Axiomatics*, World Scientific, Singapore, 2002). The concept of orthogonality, very important for quantum axiomatics, had however not yet been introduced within the formal scheme of the state property system. In this paper we introduce orthogonality in an operational way, and define ortho state property systems. Birkhoff's well known biorthogonal construction gives rise to an orthoclosure and we study the relation between this orthoclosure and the operational orthogonality that we introduced.

KEY WORDS: orthogonality; state property systems; ortho state property systems; ortho axioms.

[1] Foundations of the Exact Sciences (FUND) and Topology (TOPO), Department of Mathematics, Brussels Free University, Brussels, Belgium.
[2] The Fund for Scientific Research, Flanders, Belgium.
[3] To whom correspondence should be addressed at Foundations of the Exact Sciences (FUND) and Topology (TOPO), Department of Mathematics, Brussels Free University, Pleinlaan 2, 1050 Brussels, Belgium; e-mail: diraerts@vub.ac.be or diddesen@vub.ac.be.

1. INTRODUCTION

Within the Geneva–Brussels approach to the Foundations of Quantum Mechanics (Aerts, 1981, 1982, 1983; Piron, 1976, 1989, 1990) the basic operational concept to construct the theory is that of a "test" (in some articles also called "yes/no experiment," "question," or "operational project"). For a physical entity S one considers the set of all relevant tests Q, and denotes tests by means of symbols $\alpha, \beta, \gamma, \ldots \in Q$. The basic ontological concept is that of state of the physical entity S, and the set of all relevant states is denoted by Σ, while individual states are denoted by symbols $p, q, r, \ldots \in \Sigma$. A basic structural law on Q is the following: "if the entity S is in a state $p \in \Sigma$, such that the outcome 'yes' is certain for α, then the outcome 'yes' is certain for β." If this is satisfied we say that α implies β and denote $\alpha < \beta$. This law defines a preorder relation on Q, which induces an equivalence relation on Q. A property of the entity S is then introduced as the equivalence class of tests that test this property, the set of all relevant properties is denoted by $\mathcal{L}$ and individual properties by symbols $a, b, c, \ldots \in \mathcal{L}$. Two operations are introduced operationally on Q. For an arbitrary test $\alpha \in Q$ the "inverse test" $\widetilde{\alpha}$ is introduced, which is the test that consists of performing α and exchanging the role of "yes" and "no," and it is demanded that$\,\widetilde{}: Q \to Q$ is defined on all Q, and obviously $\widetilde{\widetilde{\alpha}} = \alpha$. For an arbitrary collection $\{\alpha_i\}$ of tests the "product test" $\Pi_i \alpha_i$ is defined as the test that consists of choosing one of the α_i and performing the chosen test and interpreting the outcome thus obtained as outcome of $\Pi_i \alpha_i$. It is also demanded that Q is closed for the product operation on tests, and as a consequence it can be proven that the set of properties $\mathcal{L}$ is a complete lattice, for the trace on $\mathcal{L}$ of the preorder relation on Q, which is a "partial order relation" on $\mathcal{L}$, denoted $<$, with the meaning: $a < b$ iff whenever the state of the entity S is such that a is actual, then also b is actual. The infimum for a collection of properties $\{a_i\}$, $a_i \in \mathcal{L}$, is denoted $\wedge_i a_i$ and it is the equivalence class of the product test $\Pi_i \alpha_i$, where for each i the test α_i tests the property a_i, hence the physical meaning of the infimum property is the conjunction.

More recently almost (and we come to this immediately) the whole scheme that is obtained as such in a purely operational way was formalized by introducing the structure of a state property system (Aerts, 1999a,b; Aerts *et al.*, 1999).

Definition 1. (State Property System). A triple $(\Sigma, \mathcal{L}, \xi)$ is called a state property system if Σ is a set, $\mathcal{L}$ is a complete lattice, and $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ is a function such that for $p \in \Sigma$, $\bar{0}$ the minimal element of $\mathcal{L}$ and $(a_i)_i \in \mathcal{L}$, we have $\bar{0} \notin \xi(p)$ (SPS1) and $a_i \in \xi(P), \forall i$ implies $\wedge_i a_i \in \xi(p)$ (SPS2). Moreover for $a, b \in \mathcal{L}$ we have that $a < b$ if and only if for every $r \in \Sigma$:$a \in \xi(r)$ implies $b \in \xi(r)$ (SPS3).

It is by the introduction of the function ξ that the state property system formalizes the operational content of the Geneva–Brussels approach. The physical meaning of $\xi(p)$ for an arbitrary state $p \in \Sigma$ of the physical entity S, is that $\xi(p)$ is the set of all properties that are actual when S is in state p. This makes it clear why (SPS1), (SPS2), and (SPS3) have to be satisfied. Indeed, (SPS1) expresses that 0, the minimal property, is the property that is never actual, for example the property "this entity S is not there." And (SPS2) expresses that the infimum of properties that are actual in a state is also an actual property, which has to be so because of the physical meaning of conjunction for the infimum. And (SPS3) expresses the physical law: $a < b$ iff whenever the state of the entity S is such that a is actual, then also b is actual.

We mentioned already that the state property system only manages to capture "almost" all of the operational structure. Indeed the structure of the inverse operation $\tilde{}: Q \rightarrow Q$, was not captured within the formal structure of the state property system. The reason why there is a fundamental problem here is because the inverse on the set of tests does not transpose to an operation on the set of properties by means of the quotient. This is because for two equivalent tests $\alpha, \beta \in Q$ we do not in general have that the inverse tests $\widetilde{\alpha}$ and $\widetilde{\beta}$ are equivalent. The problem was known in the early approaches (Aerts, 1981, 1982, 1983; Piron, 1976, 1989, 1990), and partly solved by introducing an orthogonality relation, translating part of the structure of the inverse on Q to the structure of an orthogonality relation on Σ: $p, q \in \Sigma$, then $p \perp q$ iff there exists a test $\alpha \in Q$ such that α gives with certainty "yes" if S is in state p and $\widetilde{\alpha}$ gives with certainty "yes" if S is in state q. However the structure of the inverse was in this way only transferred indirectly to a structure on $\mathcal{L}$, by demanding that two properties a and b are orthogonal iff all states that make a actual are orthogonal to all states that make b actual. A lot of the operational structure of $\tilde{}: Q \rightarrow Q$ was lost in this way.

In this article we introduce the structure of the inverse within the more complete scheme of the state property system and this will lead us to define an ortho state property system. We also want to study this "inverse" structure for the closure space that is connected through a categorical equivalence to the state property system, an equivalence of categories that has shown to be very fruitful for many other fundamental aspects of quantum axiomatics (Aerts *et al.*, 1999a,b, in press-a,b; Aerts and Deses, 2002; van der Voorde, 2000, 2001; van Steirteghem, 2000). We also introduce two "weakest" ortho axioms to make the lattice of properties of our ortho state property system to be equipped with an orthocomplementation, a necessary structure for quantum axiomatics (Aerts, 1981, 1982, 1983; Piron, 1976, 1989, 1990). Let us recall some definitions and a theorem.

Definition 2. (Cartan Map). If $(\Sigma, \mathcal{L}, \xi)$ is a state property system then its Cartan map is the mapping $\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma)$ defined by $\kappa(a) = \{p \in \Sigma \mid a \in \xi(p)\}$. This map has the property that $\kappa(\wedge_i a_i) = \cap_i \kappa(a_i)$.

Definition 3. (Closure Space). A closure space $(\Sigma, \mathcal{C})$ consists of a set Σ and a family of subsets $\mathcal{C} \subseteq \mathcal{P}(X)$, which are called closed subsets, such that $\emptyset \in \mathcal{C}$ and for $(F_i)_i \in \mathcal{C}$ we have $\cap_i F_i \in \mathcal{C}$.

Theorem 1. *If $(\Sigma, \mathcal{L}, \xi)$ is a state property system then $(\Sigma, \kappa(\mathcal{L}))$ is a closure space, called the eigenclosure of $(\Sigma, \mathcal{L}, \xi)$. Conversely, if $(\Sigma, \mathcal{C})$ is a closure space then $(\Sigma, \mathcal{C}, \bar{\xi})$ is a state property system. Here $\mathcal{C}$ is the complete lattice of closed sets, ordered by inclusion and $\bar{\xi} : \Sigma \to \mathcal{P}(\mathcal{C}) : p \mapsto \{A \in \mathcal{C} | p \in A\}$.*

For a proof of this theorem we refer to Aerts *et al.* (1999).

2. ORTHO STATE PROPERTY SYSTEMS

We are now ready to introduce the following concept of orthogonality:

Definition 4. (Ortho State Property System). An ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ is a state property system $(\Sigma, \mathcal{L}, \xi)$ and a relation $\widehat{\perp}$ on $\mathcal{L}$ such that:

$$\begin{array}{l|l} a\widehat{\perp}b \Rightarrow b\widehat{\perp}a & a_i\widehat{\perp}b_j \ \forall i, j \Rightarrow \wedge_i a_i \widehat{\perp} \wedge_j b_j \\ a\widehat{\perp}b \Rightarrow a \wedge b = \bar{0} & \qquad \bar{0}\widehat{\perp}a \qquad \forall a \in \mathcal{L} \end{array}$$

The definition of an ortho state property system is inspired by the following: if a and b are properties and there exists a test α such that α tests a and $\widetilde{\alpha}$ tests b, then the requirements of definition 4. follow. A trivial example of a $\widehat{\perp}$ relation is where we would state $a\widehat{\perp}b \Leftrightarrow a \wedge b = \bar{0}$. From now on we will assume to work with an ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$, unless explicitly stated otherwise. We can define the traditional orthogonality relation on the set of states by means of this relation $\widehat{\perp}$.

Proposition 1. *$\widehat{\perp}$ induces an orthogonality relation (antireflexive, symmetric) $\perp$ on the set of states Σ in the following way: $p \perp q$ if and only if there are $a, b \in \mathcal{L}$ such that $a\widehat{\perp}b$ and $a \in \xi(p)$ and $b \in \xi(q)$.*

Now that we have an orthogonality relation on Σ, it generates the orthoclosure $(\Sigma, \mathcal{C}_{\text{orth}})$ by means of Birkhof's biorthogonal construction: $\mathcal{C}_{\text{orth}} = \{A^{\perp\perp} | A \subset \Sigma\}$, where $A^{\perp} = \{p \in \Sigma | \forall q \in A : p \perp q\}$. Conversely an orthogonality relation on a state property system induced a $\widehat{\perp}$-relation on its property lattice, as is shown in the following proposition.

Proposition 2. *If, for a state property system $(\Sigma, \mathcal{L}, \xi)$ with an orthogonality relation $\perp$ on its states, we define $a\widehat{\perp}b$ if and only if $p \perp q \ \forall \ p, q \in \Sigma$ such that $a \in \xi(p)$ and $b \in \xi(q)$. Then $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ is an ortho state property system.*

Proof: Symmetry of $\widehat{\perp}$ is evident. The fact that $\perp$ is antireflexive implies that whenever $a\widehat{\perp}b$, we get that $a \wedge b = \bar{0}$. Since $\bar{0} \notin \xi(p)$ for any $p \in \Sigma$ we have that $p \perp q\ \forall\ p, q \in \Sigma$ such that $\bar{0} \in \xi(p)$ and $a \in \xi(q)$ is always true, hence $\bar{0}\widehat{\perp}a$ for any $a \in \mathcal{L}$. Finally $\forall i, jp \perp q\ \forall\ p, q \in \Sigma$ such that $a_i \in \xi(p)$ and $b_j \in \xi(q)$ implies $p \perp q\ \forall\ p, q \in \Sigma$ such that $\wedge_i a_i \in \xi(p)$ and $\wedge_j b \in \xi(q)$ hence we get $a_i\widehat{\perp}b_j\ \forall i, j \Rightarrow \wedge_i a_i \widehat{\perp} \wedge_j b_j$. □

3. ORTHOCOUPLES AND ORTHOPROPERTIES

There is another type of orthogonality structure that we can introduce.

Definition 5. (Orthocouple, Orthoproperty). If $a, b \in \mathcal{L}$ satisfy

$$b \in \xi(p) \Leftrightarrow p \perp q\ \forall\, q \text{ such that } a \in \xi(q)$$

$$a \in \xi(q) \Leftrightarrow q \perp p\ \forall\, p \text{ such that } b \in \xi(p)$$

they form an orthocouple. From this it follows that if a, b and a, c are orthocouples, then $b = c$. A property $a \in \mathcal{L}$ which is member of an orthocouple a, b is called an orthoproperty. For an orthoproperty $a \in \mathcal{L}$ we denote the unique property that is defined by it being member of an orthocouple by a'.

Proposition 3. *If $a, b \in \mathcal{L}$ are orthoproperties we have $(a')' = a$ and $a < b$ implies $b' < a'$.*

Proof: We have $a \in \xi(p) \Leftrightarrow p \perp q\ \forall q$ such that $a' \in \xi(q) \Leftrightarrow (a')' \in \xi(p)$. This proves that $(a')' = a$. Suppose that $a < b$ and consider $b' \in \xi(p)$. Then $p \perp q\ \forall q$ such that $b \in \xi(q)$. Since $a < b$ we also have $p \perp q\ \forall q$ such that $a \in \xi(q)$. Hence $a' \in \xi(p)$. This proves that $b' < a'$. □

The relation between the Cartan map κ and the $\perp$-relation is described as in the next propositions.

Proposition 4. *For an orthoproperty $a \in \mathcal{L}$ we have $\kappa(a') = \kappa(a)^\perp$ and $\kappa(a) = \kappa(a)^{\perp\perp}$.*

Proof: We have $p \in \kappa(a') \Leftrightarrow a' \in \xi(p) \Leftrightarrow p \perp q\ \ \forall q$ such that $a \in \xi(q) \Leftrightarrow p \perp q\ \ \forall q$ such that $q \in \kappa(a) \Leftrightarrow p \in \kappa(a)^\perp$. We remark that for $A \subset \Sigma$ we have $A^{\perp\perp\perp} = A^\perp$. From this it follows that $\kappa(a) = \kappa(a')^\perp$. Hence $\kappa(a)^{\perp\perp} = \kappa(a')^{\perp\perp\perp} = \kappa(a')^\perp = \kappa(a)$. □

Proposition 5. *A property $a \in \mathcal{L}$ is an orthoproperty iff $\kappa(a) = \kappa(a)^{\perp\perp}$ and there exists $b \in \mathcal{L}$ such that $\kappa(b) = \kappa(a)^{\perp}$ or equivalently iff $\kappa(a) \in \mathcal{C}_{\text{orth}}$ and there exists $b \in \mathcal{L}$ such that $\kappa(b) = \kappa(a)^{\perp}$. In this case $b = a'$.*

4. THE ORTHO AXIOMS

This gives us all the material that we need to put forward the first ortho axiom for an ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$.

Axiom 1. (AO1) *Axiom Ortho 1 is satisfied if there exists a generating set $\mathcal{T}$ of orthoproperties for $\mathcal{L}$, i.e., $\mathcal{L} = \{\wedge_i a_i \,|a_i \in \mathcal{T}\}$.*

The axiom to prolongate the orthocomplementation to the whole of $\mathcal{L}$ can also easily be put forward now.

Axiom 2. (AO2) *Axiom Ortho 2 is satisfied if for $p \in \Sigma$ there exists a property $a_p \in \mathcal{L}$ such that $a_p \in \xi(q) \Leftrightarrow q \perp p$. This implies the uniqueness of a_p.*

Definition 6. (Orthocomplementation). Suppose that we have a state property system $(\Sigma, \mathcal{L}, \xi)$. A function $': \mathcal{L} \to \mathcal{L}$, such that for $a, b \in$ we have:

$$\begin{array}{rl|l} (a')' = & a & a \wedge a' = \bar{0} \\ a < b \Rightarrow & b' < a' & a \vee a' = \bar{1} \end{array}$$

is called an orthocomplementation of $\mathcal{L}$.

Theorem 2. *We have:*

(A) If an ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ satisfies AO1 and AO2 then it induces an orthocomplementation $' : \mathcal{L} \to \mathcal{L}$ on the state property system $(\Sigma, \mathcal{L}, \xi)$. Here a' is defined as the unique member of $\mathcal{L}$ for which a, a' forms an orthocouple in $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$.

(B) If a state property system $(\Sigma, \mathcal{L}, \xi)$ has an orthocomplementation $' : \mathcal{L} \to \mathcal{L}$ then $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ is an ortho state property system satisfying AO1 and AO2, where $\widehat{\perp}$ is defined by $a \widehat{\perp} b \Leftrightarrow b < a'$.

Proof: (A): We prove that for $a \in \mathcal{L}$ there exists a unique property a' such that: $a' \in \xi(p) \Leftrightarrow p \perp q \; \forall q \in \Sigma$ such that $a \in \xi(q)$. The fact that for each $p \in \Sigma$ AO2 gives us a unique $a_p \in \mathcal{L}$, allows us to define $a' = \wedge_{a \in \xi(p)} a_p$. Suppose that $a' \in \xi(r)$ for some $r \in \Sigma$ then obviously for a $p \in \Sigma$ such that $a \in \xi(p)$ we get $a' < a_p$. Hence $a_p \in \xi(r)$, for each $p \in \Sigma$ such that $a \in \xi(p)$. Therefore $r \perp p$ for each $p \in \Sigma$ such that $a \in \xi(p)$. Conversely, if $r \perp q$ for every $q \in \Sigma$ such that $a \in \xi(q)$ we know from AO2 that $a_q \in \xi(r)$ for every such q. Hence $a' = \wedge_{a \in \xi(q)} a_q \in \xi(r)$.

By the last two results we have that our a' is the same as a' defined by Definition 5. There remains to prove that $'$ is indeed an orthocomplementation. For $a \in \mathcal{L}$ we know that there exists orthoproperties a_i such that $a = \wedge_i a_i$, for which $(a_i')' = a_i$. Thus $(a')' \in \xi(r)$ which is equivalent to

$$
\begin{aligned}
& r \perp q \text{ for every } q \in \Sigma \text{ such that } a' \in \xi(q) \\
\Leftrightarrow\ & r \perp q \text{ for every } q \in \Sigma \text{ such that } q \perp p \text{ for every } p \in \Sigma \text{ such that } a \in \xi(p) \\
\Leftrightarrow\ & \forall i : r \perp q \text{ for every } q \in \Sigma \text{ such that } q \perp p \text{ for every } p \in \Sigma \text{ such that } \\
& a_i \in \xi(p) \\
\Leftrightarrow\ & \forall i : r \perp q \text{ for every } q \in \Sigma \text{ such that } a_i' \in \xi(q) \\
\Leftrightarrow\ & \forall i : (a_i')' = a_i \in \xi(r) \\
\Leftrightarrow\ & a \in \xi(r)
\end{aligned}
$$

Hence $(a')' = a$. If $a < b$ and $b' \in \xi(r)$ we have that $r \perp q$ for every $q \in \Sigma$ such that $b \in \xi(q)$, but $a \in \xi(q)$ implies $b \in \xi(q)$ hence $r \perp q$ for every $q \in \Sigma$ such that $a \in \xi(q)$, so $a' \in \xi(r)$ and thus $b' < a'$. Finally, if $\bar{0} \neq a \wedge a' \in \xi(r)$ then $a, a' \in \xi(r)$ but this implies $r \perp r$ which is impossible, so $a \wedge a' = \bar{0}$. Analogously $a \vee a' = \bar{1}$.

(B): We only give the proof of the last condition on $\widehat{\perp}$, the others are easy verifications. Let $a_i \widehat{\perp} b_j$ for every i, j, then by definition of $\widehat{\perp}$ we get $b_j < a_i'$, so $b_j < \wedge_i a_i'$. We also have $\wedge_i a_i < a_i$, hence $a_i' < (\wedge_i a_i)'$ so that $\wedge_i a_i' < (\wedge_i a_i)'$. Therefore $b_j < (\wedge_i a_i)'$ and thus $\wedge_j b_j < (\wedge_i a_i)'$. Finally we conclude that $\wedge_i a_i \widehat{\perp} \wedge_j b_j$. In order to prove AO1 we will prove that every pair a, a' is an orthocouple, i.e.:

$$
\begin{aligned}
a \in \xi(q) &\Leftrightarrow q \perp p\ \forall\, p \text{ such that } a' \in \xi(p) \\
a' \in \xi(p) &\Leftrightarrow\ p \perp q\ \forall\, q \text{ such that } a \in \xi(q)
\end{aligned}
$$

where $\perp$ is given by $p \perp q \Leftrightarrow \exists a, b \in \mathcal{L} : a \widehat{\perp} b, a \in \xi(p), b \in \xi(q)$. We prove the first statement, the second is completely analogous. Let $a \in \xi(q)$ and p such that $a' \in \xi(p)$ then obviously there $a \widehat{\perp} a'$, hence $p \perp q$. Conversely, suppose $p \perp q$ for each p such that $a' \in \xi(p)$. So for such a p there are $\tilde{a} \in \xi(q), \tilde{b} \in \xi(p)$ for which $\tilde{a} \widehat{\perp} \tilde{b}$, so $\tilde{b} < \tilde{a}'$. Since $\tilde{b} \in \xi(p)$, we know that $\tilde{a}' \in \xi(p)$. We have:

$$
\forall p \text{ such that } a' \in \xi(p) : \exists \tilde{a}_p \in \xi(q) : \tilde{a}_p' \in \xi(p)
$$

We now define $c = \wedge_{a' \in \xi(p)} \tilde{a}_p$. Since $c < \tilde{a}_p$, we have that $\tilde{a}_p' < c'$, hence $c' \in \xi(p)$, for each p such that $a' \in \xi(p)$. This means that $a' < c'$. Using the orthocomplementation we get $c < a$, but since $c \in \xi(q)$, $a \in \xi(q)$. Hence AO1 holds. From the above we also see that the orthocomplementation induced by the ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$, is the same as the given one, since a, a'

always form an orthocouple. In order to prove AO2 we choose an $p \in \Sigma$ and consider $a_p = (\wedge\xi(p))'$. If $a_p \in \xi(q)$ then for $\tilde{a} = \wedge\xi(p) \in \xi(p)$ and $\tilde{b} = a_p \in \xi(q)$ we have that $\tilde{b} < \tilde{a}'$, hence $\tilde{a}\widehat{\perp}\tilde{b}$ and thus $p \perp q$. Conversely, if $p \perp q$ then there are $\tilde{a} \in \xi(p), \tilde{b} \in \xi(q)$ for which $\tilde{b} < \tilde{a}'$. Since $a'_p = \wedge\xi(p) < \tilde{a}$ we know that $\tilde{a}' < a_p$, so $\tilde{b} < a_p$ which implies $a_p \in \xi(q)$. Hence AO2 also holds and we have proven the theorem. $\square$

From the proof of this theorem one has the following corollary which we shall need further on.

Corollary 1. *Take an ortho state property system* $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ *for which AO1 and AO2 are satisfied. Let* $' : \mathcal{L} \to \mathcal{L}$ *be the orthocomplementation described in the theorem, then* $a' = \wedge_{a\in\xi(p)} a_p$ *where the* a_p *are given by AO2. Moreover every property of* $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ *is an orthoproperty.*

5. EIGENCLOSURE AND ORTHOCLOSURE

The previous theorem describes the link between an ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ and an orthocomplementation $' : \mathcal{L} \to \mathcal{L}$. In what follows we'll turn our attention toward the associated closure spaces: the eigenclosure and the orthoclosure.

Theorem 3. *Consider an ortho state property system* $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$*, then:*

$$\kappa(\mathcal{L}) = \mathcal{C}_{\text{orth}} \Leftrightarrow \text{ AO1 and AO2}$$

Proof: Let A be closed in $(\Sigma, \mathcal{C}_{\text{orth}})$, i.e., $A = A^{\perp\perp}$. By AO2 we know that $\forall p \in A : \exists a_p \in \mathcal{L} : a_p \in \xi(q) \Leftrightarrow q \perp p$. We make $a = \wedge\{a_p | p \in A\}$ and define $A^* = \kappa(a)$. Then $q \in A^*$ is equivalent to $q \in \kappa(\wedge\{a_p | p \in A\}) = \cap_{p\in A}\kappa(a_p)$. So for any $p \in A$ one has that $q \in \kappa(a_p)$, which means that $a_p \in \xi(q)$. Using (AO2) we obtain that $q \in A^*$ is equivalent to $p \perp q$ for every $p \in A$, so $q \in A^{\perp\perp}$ Hence $A^* = A^{\perp\perp} = \kappa(a)$ which is closed in $(\Sigma, \kappa(\mathcal{L}))$. Conversely, if A is closed in $(\Sigma, \kappa(\mathcal{L}))$, there is an $a \in \mathcal{L}$ such that $A = \kappa(a)$. Since a is an orthoproperty we know that $\kappa(a) = \kappa(a)^{\perp\perp}$ so $A = A^{\perp\perp}$ is closed in $(\Sigma, \mathcal{C}_{\text{orth}})$.

Let $p \in \Sigma$. $\{p\}^\perp \in \mathcal{C}_{\text{orth}}$ since $\{p\}^{\perp\perp\perp} = \{p\}^\perp$ since $\kappa(\mathcal{L}) = \mathcal{C}_{\text{orth}}$ there is a property a such that $\kappa(a) = \{p\}^\perp$. For this a we have the following chain of equivalences $a \in \xi(q) \Leftrightarrow q \in \kappa(a) \Leftrightarrow q \in \{p\}^\perp \Leftrightarrow q \perp p$. So for any p there is an $a = a_p$ such that $a \in \xi(q) \Leftrightarrow q \perp p$, hence AO2 follows. Clearly $\mathcal{T} = \mathcal{L}$ is a generating set. Take $a \in \mathcal{L}$ then $\kappa(a) \in \kappa(\mathcal{L}) = \mathcal{C}_{\text{orth}}$, so $\kappa(a) = \kappa^{\perp\perp}$. By Definition of $\mathcal{C}_{\text{orth}}$ we know that $\kappa(a)^\perp \in \mathcal{C}_{\text{orth}} = \kappa(\mathcal{L})$, so there is a $b \in \mathcal{L}$ such that $\kappa(b) = \kappa(a)^\perp$. By Proposition 5 we see that a is an orthoproperty, hence AO1 also holds. $\square$

Theorem 4. *We have:*

(C) If an ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ satisfies AO1 and AO2 then the closure space $(\Sigma, \kappa(\mathcal{L}))$ is induced by the underlying $\perp$-relation of Proposition 1.

(D) If a closure space $(\Sigma, \mathcal{C})$ is induced by a $\perp$-relation then $(\Sigma, \mathcal{C}, \bar{\xi}, \widehat{\perp})$ is an ortho state property system satisfying AO1 and AO2, where $\widehat{\perp}$ is as in Proposition 2.

Proof: (C): Follows from the previous Theorem 3 and Proposition 1.

(D): Let $\mathcal{C}$ be induced by an orthogonality relation $\perp$. From Proposition 2 we know that $(\Sigma, \mathcal{C}, \bar{\xi}, \widehat{\perp})$ is an ortho state property system. Let us denote by $\perp^*$ the underlying $\perp$-relation, i.e. $p \perp^* q \Leftrightarrow \exists A, B \in \mathcal{C} : A \widehat{\perp} B, p \in A, q \in B$. Suppose that $p \perp^* q$, then there are $A, B \in \mathcal{C}$ with $p \in A$ and $q \in B$ such that $A \widehat{\perp} B$. Hence for any $\tilde{p} \in A$ and $\tilde{q} \in B$ we have that $\tilde{p} \perp \tilde{q}$, so since $p \in A$ and $q \in B$ we get that $p \perp q$. Conversely, if $p \perp q$, then we choose $A = \{p\}^{\perp\perp}$ and $B = \{q\}^{\perp\perp}$. Obviously $p \in A$ and $q \in B$, moreover if $\tilde{p} \in A$ and $\tilde{q} \in B$ then $\tilde{p} \perp r$ for any r such that $r \perp p$. In particular for $r = q$ we have $\tilde{p} \perp q$, hence $\tilde{p} \in \{q\}^{\perp}$, so $\tilde{q} \perp \tilde{p}$. Thus $A \widehat{\perp} B$. Finally we have: $\exists A, B \in \mathcal{C} : A \widehat{\perp} B, p \in A, q \in B$, so we conclude that $p \perp^* q$. By the equivalence between closure spaces and state property systems we know that $\kappa(\mathcal{C}) = \mathcal{C} = \mathcal{C}_{\text{orth}}$, hence by Theorem 3 we have that AO1 and AO2 are fulfilled. □

With the above results (A), (B), (C), and (D) we consider the following scheme. We start with an ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ satisfying AO1 and AO2. First we use (A) to get a state property system $(\Sigma, \mathcal{L}, \xi)$ and an orthocomplementation $' : \mathcal{L} \to \mathcal{L}$. Applying (B) we get a new ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp}^*)$, which satisfies AO1 and AO2. On the other hand we can apply (C) to the ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$, hence we get a closure space $(\Sigma, \kappa(\mathcal{L}))$ where $\kappa(\mathcal{L}) = \mathcal{C}_{\text{orth}}$ is induced by the orthogonality relation $\perp$ of $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$. Using (D) we get again an ortho state property system $(\Sigma, \kappa(\mathcal{L}), \bar{\xi}, \widehat{\perp}^{**})$, satisfying AO1 and AO2. We now ask ourselves what the relation is between those three ortho state property systems. First we note that by the general equivalence between state property systems and closure spaces $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma, \kappa(\mathcal{L}), \bar{\xi})$ can be considered as being the same (up to isomorphism). The relation between $\widehat{\perp}^*$ and $\widehat{\perp}^{**}$ is given in the following theorem.

Theorem 5. *With the above notations, we have $\kappa(a)\widehat{\perp}^{**}\kappa(b) \Leftrightarrow a\widehat{\perp}^{*}b$.*

Proof: $\kappa(a)\widehat{\perp}^{**}\kappa(b)$ is equivalent to $p \perp q, \forall p, q$ such that $a \in \xi(p)$ and $b \in \xi(q)$. Thus, for any p such that $a \in \xi(p)$, one has that $b \in \xi(q)$ implies $p \perp q$. By means of (AO2) it also implies $b < a_p$, hence $\kappa(a)\widehat{\perp}^{**}\kappa(b)$ is equivalent to $b < \wedge_{a \in \xi(p)} a_p = a'$, which means that $a\widehat{\perp}^{*}b$. □

From this we know that $(\Sigma, \mathcal{L}, \xi, \widehat{\perp}^*)$ and $(\Sigma, \kappa(\mathcal{L}), \bar{\xi}, \widehat{\perp}^{**})$ are essentially the same. In order to compare $(\Sigma, \mathcal{L}, \xi, \widehat{\perp}^*)$ with the original ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ we need one more proposition.

Proposition 6. *For any ortho state property system* $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$ *satisfying AO1 and AO2 we have* $a \widehat{\perp} b \Rightarrow b < a'$.

Proof: $a \widehat{\perp} b$ implies $p \perp q$, $\forall p, q$ such that $a \in \xi(p)$ and $b \in \xi(q)$. With the same reasoning as in the previous proof one finds that it also implies $b < a'$. □

Theorem 6. $\widehat{\perp}^*$ *is the largest relation such that* $(\Sigma, \mathcal{L}, \xi, \widehat{\perp}^*)$ *is an ortho state property system, satisfying AO1 and AO2, with the same orthocomplementation* $' : \mathcal{L} \to \mathcal{L}$ *as* $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$.

Proof: Consider a relation $\tilde{\widehat{\perp}}$ such that $(\Sigma, \mathcal{L}, \xi, \tilde{\widehat{\perp}})$ is an ortho state property system, satisfying AO1 and AO2, with the same orthocomplementation $' : \mathcal{L} \to \mathcal{L}$ as $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$. By the previous proposition we have $a \tilde{\widehat{\perp}} b$ which implies $b < a'$. Therefore $a \widehat{\perp}^* b$ and thus $\tilde{\widehat{\perp}} \subset \widehat{\perp}^*$. □

To conclude we give an example showing that $\widehat{\perp}^*$ can be strictly larger than $\widehat{\perp}$.

Example 1. Consider a set of states $\Sigma = \{p, q, r, s, t, u\}$ and the property lattice $\mathcal{L}$ (see Fig. 1), with top $\bar{1} = 10$ and bottom $\bar{0} = 1$.

We define the map ξ by $\xi(p) = \{3, 6, 7, 9, 10\}$, $\xi(q) = \{2, 4, 5, 8, 10\}$, $\xi(r) = \{6, 9, 10\}$, $\xi(s) = \{5, 8, 10\}$, $\xi(t) = \{7, 9, 10\}$, $\xi(u) = \{4, 8, 10\}$. In this way $(\Sigma, \mathcal{L}, \xi)$ is a state property system. We endow it with the following relation:

$$\widehat{\perp} = \{(i, 1), (1, i) | 1 \leq i \leq 10\} \cup \{(7, 5), (5, 7), (4, 6), (6, 4)\}$$

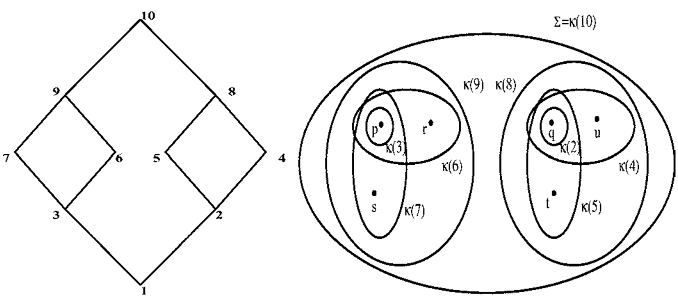

Fig. 1. The lattice $\mathcal{L}$ and the closure space $\kappa(\mathcal{L})$.

Hence we get an ortho state property system $(\Sigma, \mathcal{L}, \xi, \widehat{\perp})$. Since $\mathcal{C}_{\text{orth}} = \kappa(\mathcal{L})$ we have that both AO1 and AO2 are satisfied. We can now consider $\widehat{\perp}^*$. Since $\kappa(2) = \{q\}, \kappa(3) = \{p\}$ and $p \perp q$, we have that $\kappa(2)\widehat{\perp}^{**}\kappa(3)$, hence $2\widehat{\perp}^*3$. So $\widehat{\perp}^*$ is strictly larger than $\widehat{\perp}$. In fact it is given by:

$$\widehat{\perp}^* = \widehat{\perp} \cup \{(3, 2), (2, 3), (4, 3), (3, 4), (3, 5), (2, 7),$$
$$(2, 6), (7, 2), (6, 2), (2, 9), (9, 2), (5, 3), (3, 8), (8, 3)\}$$

REFERENCES

Aerts, D. (1981). *The One and the Many: Towards a Unification of the Quantum and the Classical Description of One and Many Physical Entities*, Doctoral Thesis, Brussels Free University, Brussels, Belgium.

Aerts, D. (1982). Description of many physical entities without the paradoxes encountered in quantum mechanics. *Foundations of Physics* **12**, 1131–1170.

Aerts, D. (1983). Classical theories and nonclassical theories as a special case of a more general theory. *Journal of Mathematical Physics* **24**, 2441–2453.

Aerts, D. (1999a). Foundations of quantum physics: A general realistic and operational approach. *International Journal of Theoretical Physics* **38**, 289–358.

Aerts, D. (1999b). Quantum mechanics: Structures, axioms, and paradoxes. In *Quantum Mechanics and the Nature of Reality*, D. Aerts and J. Pykacz, eds., Kluwer Academic, Dordrecht, The Netherlands, lanl archive ref and link: quant-ph/0106132.

Aerts, D., Colebunders, E., van der Voorde, A., and van Steirteghem, B. (1999). State property systems and closure spaces: A study of categorical equivalence. *International Journal of Theoretical Physics* **38**, 359–385.

Aerts, D., Colebunders, E., van der Voorde, A., and van Steirteghem, B. (in press-a). The construct of closure spaces as the amnestic modification of the physical theory of state property systems, *Applied Categorical Structures*.

Aerts, D. and Deses, D. (2002). State property systems and closure spaces: Extracting the classical and nonclassical parts. In *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Computation, and Axiomatics*, D. Aerts, M. Czachor, and T. Durt, eds., World Scientific, Singapore.

Aerts, D., van der Voorde, A., and Deses, D. (2001). Connectedness applied to closure spaces and state property systems. *Journal of Electrical Engineering* **52**, 18–21.

Aerts, D., van der Voorde, A., and Deses, D. (in press-b). Classicality and connectedness for state property systems and closure spaces, *International Journal of Theoretical Physics*.

Piron, C. (1976). *Foundations of Quantum Physics*, W. A. Benjamin, Reading, MA.

Piron, C. (1989). Recent developments in quantum mechanics. *Helvetica Physica Acta* **62**, 82.

Piron, C. (1990). *Mécanique Quantique: Bases et Applications*, Press Polytechnique de Lausanne, Switzerland.

Van der Voorde, A. (2000). A categorical approach to T_1 separation and the product of state property systems. *International Journal of Theoretical Physics* **39**, 947–953.

Van der Voorde, A. (2001). *Separation Axioms in Extension Theory for Closure Spaces and Their Relevance to State Property Systems*, Doctoral Thesis, Brussels Free University, Brussels, Belgium.

Van Steirteghem, B. (2000). T_0 separation in axiomatic quantum mechanics. *International Journal of Theoretical Physics* **39**, 955.